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On the Path Integral Treatment for an Aharonov–Bohm Field on the Hyperbolic Plane

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In this paper I discuss by means of path integrals the quantum dynamics of a charged particle on the hyperbolic plane under the influence of an Aharonov–Bohm gauge field. The path integral can be solved in terms of an expansion of the homotopy classes of paths. I discuss the interference pattern of scattering by an Aharonov–Bohm gauge field in the flat-space limit, yielding a characteristic oscillating behavior in terms of the field strength. In addition, the cases of the isotropic Higgs oscillator and the Kepler–Coulomb potential on the hyperbolic plane are briefly sketched.

1. INTRODUCTION

The Aharonov–Bohm gauge field has a long history, beginning in 1959 with a classical paper by Aharonov and Bohm (1959). The effect has been well studied and well confirmed (Anandan and Safko, 1994), but not necessarily well understood. It describes the motion of charged particles, i.e., electrons, which are scattered by an infinitesimal thin solenoid. The magnetic vector potential $\mathbf{A}$ of the solenoid produces a magnetic field which is essentially δ-like, i.e., its support is an infinitesimal thin solenoid, and it is vanishing everywhere else. Geometrically this experimental setup corresponds to the quantum motion of a particle (which we consider as spinless) in $\mathbb{R}^2$, where a point has been removed with the consequence that topologically $\mathbb{R}^2$ becomes no longer connected. Since the solenoid is assumed impenetrable, the space of the particle motion $\mathbb{M}$ is the Euclidean plane minus the cross section of the solenoid. Everywhere in $\mathbb{M}$, $\nabla \times \mathbf{A} = 0$ and hence $\mathbf{A} = \nabla f(r)$, where $f(r)$ is an arbitrary scalar function of $r = |\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^2$. Classically, a charged particle is not affected at all by the solenoid. However, in quantum mechanics,

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the particle's wavefunction picks up in a scattering experiment a phase factor according to

$$\Psi_\alpha(\mathbf{x}) = \Psi_0(\mathbf{x}) \exp\left(\frac{ie}{\hbar c}\int_{\text{path }\alpha} \mathbf{A}\cdot d\mathbf{x}\right) \tag{1}$$

where $\Psi_0(\mathbf{x})$ is the vector potential-free solution. The wavefunction Ψ effective to a measurement is the sum of solutions corresponding to inequivalent paths, i.e., $\Psi = \Sigma_\alpha \Psi_\alpha$. Topologically the paths α can be distinguished by their winding numbers n, thus giving rise to infinitely many homotopy classes designated by the number n.

Path integral treatments of the Aharonov–Bohm effect in the Euclidean plane are due to Bernido and Inomata (1980, 1981), Gerry and Singh (1979, 1982, 1983), Liang (1988), and Schulman (1971). Harmonic interactions have been dealt with by Kibler and Campigotto (1993), the Coulomb–Kepler potential has been taken into account by Chetounai *et al.* (1989), Drăgănascu *et al.* (1992), Hoang *et al.* (1992), Kibler and Negadi (1987), Lin (1998), and Park and Yoo (1998), and relativistic particles by, e.g., Bernido (1993), Gamboa and Rivelles (1991), Hoang *et al.* (1992), Hoang and Giang (1993), Lin (1998), and Park and Yoo (1998); a more comprehensive bibliography can be found in, e.g., Anandan and Safko (1994), or Grosche and Steiner (1998).

Feynman path integrals (e.g., Feynman and Hibbs, 1965; Grosche, 1996; Grosche and Steiner, 1998; Kleinert, 1995; Schulman, 1981) provide us with global information on the quantum motion, including the topological effects on the wavefunction. If we want to study the Aharonov–Bohm effect by means of path integrals (Berndio and Inomata, 1980, 1981; Gerry and Singh, 1979, 1982, 1983; Liang, 1988), we consider the time evolution form $t = 0$ to $t = T$ of the wavefunction of a particle according to

$$\Psi_\alpha(\mathbf{x}''; T) = \sum_\beta \int K_{\alpha\beta}(\mathbf{x}'', \mathbf{x}'; T)\Psi_\alpha(\mathbf{x}'; 0)\, d\mathbf{x}' \tag{2}$$

where

$$K_{\alpha\beta}(\mathbf{x}'', \mathbf{x}'; T) = K_0(\mathbf{x}'', \mathbf{x}'; T) \exp\left[\frac{ie}{\hbar c}\left(\int_{\text{path }\alpha}^{\mathbf{x}''} - \int_{\text{path }\beta}^{\mathbf{x}'}\right)\mathbf{A}\cdot d\mathbf{x}\right] \tag{3}$$

and this leads us to the formal expression separating the sum over α and β (under the assumption the separation is well defined)

$$\sum_{\alpha,\beta} K_{\alpha\beta}\Psi_\beta = K\sum_\beta \Psi_\beta \tag{4}$$

Provided the paths α, β cover in an idealized experiment the whole range from minus infinity to plus infinity, we can express the separation of the time evolution of the particle according to

$$K(\mathbf{x}'', \mathbf{x}'; T) = \sum_{n=-\infty}^{\infty} K_n(\mathbf{x}'', \mathbf{x}'; T) \tag{5}$$

where $n = 0$ denotes the unperturbed case in $\mathbb{R}^2$, i.e., we obtain the free propagator on the entire $\mathbb{R}^2$. For the final result we obtain for the Feynman kernel the following form (e.g., Berndio and Inomata, 1980, 1981; Grosche and Steiner, 1998; Liang, 1988):

$$K(\mathbf{x}'', \mathbf{x}'; T) = \frac{m}{2\pi i\hbar T} \exp\left(\frac{im}{2\hbar T}(r'^2 + r''^2)\right) \sum_{n=-\infty}^{\infty} e^{in(\varphi''-\varphi')} I_{|n-\xi|}\left(\frac{mr'r''}{i\hbar T}\right) \tag{6}$$

Here, two-dimensional polar coordinates (r, φ) have been used, and $\xi = e\Phi/2\pi\hbar c$ with $\Phi = B \times$ area the magnetic flux.

2. AHARONOV–BOHM FIELD ON THE HYPERBOLIC PLANE

In this paper I would like to give a path integral treatment of the Aharonov–Bohm effect on the hyperbolic plane (Kuperin *et al.*, 1994), i.e., the scattering of (spinless) electrons by an Aharonov–Bohm field on leaky tori. Such systems play an important role in the theory of quantum chaos (e.g., Gutzwiller, 1991). The hyperbolic plane, respectively Lobachevsky space, is defined as one sheet of the double-sheeted hyperboloid

$$\mathbf{u}^2 = u_0^2 - u_1^2 - u_2^2 = R^2, \qquad u_0 > 0 \tag{7}$$

The model of the upper-half plane $U = \{\Im(z) = y > 0 | z = x + iy\}$ endowed with the metric has the form (where I have set for simplicity $R = 1$)

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \qquad x \in \mathbb{R}, \quad y > 0 \tag{8}$$

Alternatively we can also consider the unit-disk model $D = \{z = re^{i\vartheta} | r < 1, \vartheta \in [0, 2\pi)\}$

$$ds^2 = 4\frac{dr^2 + r^2\, d\vartheta^2}{(1 - r^2)^2}, \qquad r < 1, \quad \vartheta \in [0, 2\pi) \tag{9}$$

and the pseudosphere $\Lambda = \{z = i \tanh(\tau/2)\, e^{-i\varphi} | \tau > 0, \varphi \in [0, 2\pi)\}$

$$ds^2 = d\tau^2 + \sinh^2\tau\, d\varphi^2, \qquad \tau > 0, \quad \varphi \in [0, 2\pi) \tag{10}$$

U, D, and Λ are three coordinate-space representations out of nine of the hyperbolic plane (Grosche *et al.*, 1996; Grosche, 1996; Olevskiı̆, 1950). Plane waves have the asymptotic representation $\propto y^{1/2 \pm ik}$ (e.g., on U, k the wavenumber), $e^{-(\pm ik + 1/2)\tau}$ (on Λ), and the coordinate origin is $r = 0$ (on D), $\tau = 0$ (on Λ), and $z = i$ (on U), respectively. The isometries on the hyperbolic plane are Möbius transformations corresponding to the symmetry group PSL(2, $\mathbb{R}$), and magnetic fields give rise to the consideration of automorphic forms in the theory of the Selberg trace formula (Hejhal, 1976).

Constant magnetic fields on the hyperbolic plane have been studied by, e.g., Comtet (1987), Fay (1977), and Pnueli (1994), and by means of path integrals by Grosche (1988, 1990a). The path integral formulation for a particle on the hyperbolic plane subject to a constant magnetic field on Λ has the form (Grosche, 1990a) (I implicitly assume that the constant negative curvature of the hyperbolic plane, i.e., the two-dimensional hyperboloid, equals one, $\mathbf{u} \in \Lambda$)

$$
\begin{aligned}
&K(\mathbf{u}'', \mathbf{u}'; T) \\
&\quad \equiv K(\tau'', \tau', \varphi'', \varphi'; T) \\
&\quad = \int_{\tau(0)=\tau'}^{\tau(T)=\tau''} \mathcal{D}\tau(t) \sinh\tau \int_{\varphi(0)=\varphi'}^{\varphi(T)=\varphi''} \mathcal{D}\varphi(t) \\
&\qquad \times \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m}{2}(\dot\tau^2 + \sinh^2\tau\,\dot\varphi^2) - b(\cosh\tau - 1)\dot\varphi \right.\right. \\
&\qquad\qquad \left.\left. - \frac{\hbar^2}{8m}\left(1 - \frac{1}{\sinh^2\tau}\right)\right] dt\right\} \\
&\quad = \exp\left(-\frac{i\hbar T}{8m}\right) \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\epsilon}\right)^N \prod_{j=1}^{N-1} \int_0^\infty \sinh\tau_j\, d\tau_j \int_0^{2\pi} d\varphi_j \\
&\qquad \times \exp\left[\frac{i}{\hbar}\sum_{j=1}^N \left(\frac{m}{2\epsilon}(\Delta^2\tau_j + \widehat{\sinh^2}\tau_j\,\Delta^2\varphi_j) \right.\right. \\
&\qquad\qquad \left.\left. - b(\widehat{\cosh}\,\tau_j - 1)\Delta\varphi_j - \frac{\epsilon\hbar^2}{8m\,\sinh^2\tau_j}\right)\right] \\
&\quad = \sum_{l=-\infty}^{\infty} \left[\sum_{N=0}^{N_{\max}} e^{-iE_N T/\hbar}\, \Psi_{Nl}^{b}(\tau'', \varphi'')\Psi_{Nl}^{b*}(\tau', \varphi') \right. \\
&\qquad \left. + \int_0^\infty dK\, e^{-iE_k T/\hbar}\, \Psi_{kl}^{b}(\tau'', \varphi'')\Psi_{kl}^{b*}(\tau', \varphi')\right]
\end{aligned}
\tag{11}
$$

Here $b = eB/\hbar c$, with B the strength of the magnetic field; c denotes the velocity of light. For the magnetic field $\mathbf{B}$ I have chosen the gauge

$$\mathbf{A} = \begin{pmatrix} A_\tau \\ A_\varphi \end{pmatrix} = B(\cosh\tau - 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{12}$$

Due to $dB = (\partial_\tau A_\varphi - \partial_\varphi A_\tau)\, d\tau \wedge d\varphi = (m/2)B \sinh\tau\, d\tau \wedge d\varphi$, dB has the form *constant* × *volume form* and can thus be interpreted indeed as a constant magnetic field. In the lattice formulation I have taken (Grosche, 1996; Grosche and Steiner, 1998) $\Delta q_j = q_j - q_{j-1}$, $q_j = q(t_j)$, $t_j = j\epsilon$, $j = 1, \dots, N$, $\epsilon = T/N$, $N \to \infty$, $f^2(q_j) \equiv f(q_{j-1})f(q_j)$, for any function f of the coordinates. The bound-state solutions are given by

$$\Psi^b_{N,l}(\tau, \varphi) = \left[\frac{N!\,(2b + |l|)\Gamma(2b - N + |l|)}{4\pi(N + |l|)!\,\Gamma(2b - N)}\right]^{1/2} \times e^{il\varphi}\left(\tanh\frac{\tau}{2}\right)^{|l|}\left(1 - \tanh^2\frac{\tau}{2}\right)^{b-N} \times P_N^{(|l|, 2b-2N-1)}\left(1 - 2\tanh^2\frac{\tau}{2}\right) \tag{13}$$

$$E_N = \frac{\hbar^2}{2m}\left[b^2 + \frac{1}{4} - \left(b - N - \frac{1}{2}\right)^2\right], \qquad \left(N = 0, 1, \dots \le N_{\max} < b - \tfrac{1}{2}\right) \tag{14}$$

$P_n^{(a,b)}(x)$ are Jacobi polynomials (Gradshteyn and Ryzhik, 1980). The energy levels (14) are the Landau levels on the hyperbolic plane. This is in complete analogy to the flat-space case, where the Landau levels are $E_n = \hbar\omega(n + \frac{1}{2})$ with $\omega = eB/\hbar c$ the cyclotron frequency, and the bound states are described by Laguerre polynomials (e.g., Grosche and Steiner, 1998). The flat-space limit can be recovered (Grosche *et al.*, 1996) by reintroducing the constant curvature $k = 1/R$ $(R > 0)$, redefining $E_N \to E_N/R^2$, $b \to bR^2$ [note $b(\cosh\tau - 1) \to br^2R^2/2$, $r > 0$ the polar variable in $\mathbb{R}^2$, as $R \to \infty$], and considering the limit $R \to \infty$.

For the continuous states, the wavefunctions and the energy spectrum, respectively, are

$$\Psi^b_{k,l}(\tau,\varphi) = \frac{1}{\pi|l|!}\sqrt{\frac{k\sinh 2\pi k}{4\pi}}\,\Gamma\left(\frac{1+ik}{2}+b+|l|\right)\Gamma\left(\frac{1+ik}{2}-b\right)$$
$$\times e^{il\varphi}\left(\tanh\frac{\tau}{2}\right)^{|l|}\left(1-\tanh^2\frac{\tau}{2}\right)^{1/2+ik}$$
$$\times {}_2F_1\left(\frac{1}{2}-ik+b+|l|, \frac{1}{2}+ik-b; 1+|l|; \tanh^2\frac{\tau}{2}\right) \tag{15}$$

$$E_k = \frac{\hbar^2}{2m}\left(k^2+b^2+\frac{1}{4}\right) \tag{16}$$

${}_2F_1(a, b; c; z)$ is the hypergeometric function, and $k > 0$ denotes the wavenumber. I note that a minimum strength of B is required in order that bound states can occur, and only a finite number of bound states can exist. For the case that the magnetic field vanishes we obtain (Grosche and Steiner, 1988) [for the relation of the Legendre functions to the hypergeometric function see, e.g., Gradshteyn and Ryzhik (1980)]

$$\Psi_{k,l} = \sqrt{\frac{k\sinh\pi k}{2\pi^2}}\,\Gamma(\tfrac{1}{2}+ik+|l|)\,e^{il\varphi}\mathcal{P}^{-|l|}_{ik-1/2}(\cosh\tau) \tag{17}$$

$$E_k = \frac{\hbar^2}{2m}\left(k^2+\frac{1}{4}\right) \tag{18}$$

For instance, we have the relation (Abramowitz and Stegun, 1984)

$$\mathcal{P}^{\mu}_{\nu-1/2}(\cosh\tau) = \frac{1}{\Gamma(1-\mu)}\,2^{2\mu}(1-e^{-2\tau})^{-\mu}\,e^{-(\nu+1/2)\tau}$$
$$\times {}_2F_1\left(\frac{1}{2}-\mu; \frac{1}{2}+\nu-\mu; 1-2\mu; 1-e^{-2\tau}\right) \tag{19}$$

However, for the vector potential for an Aharonov–Bohm gauge field, we need another Ansatz. Following Kuperin *et al.* (1994), I take $\mathbf{A} = B\mathbf{e}_\varphi$ with $B = \text{const}$. Therefore we get for the classical Hamiltonian

$$\mathcal{H} = \frac{\hbar^2}{2m}\left[p_\tau^2 + \frac{1}{\sinh^2\tau}\left(p_\varphi - \frac{eB}{\hbar c}\right)^2\right] \tag{20}$$

and for the Lagrangian ($b = eB/\hbar c$)

$$\mathcal{L} = \frac{m}{2}(\dot\tau^2+\sinh^2\tau\,\dot\varphi^2) + \frac{e}{c}\mathbf{A}\cdot\begin{pmatrix}\dot\tau\\ \dot\varphi\end{pmatrix} = \frac{m}{2}(\dot\tau^2+\sinh^2\tau\,\dot\varphi^2)+\xi\dot\varphi \tag{21}$$

Note that the vector potential in (12) vanishes at $\tau = 0$, which means that we can take any constant for A_φ depending on the gauge, and the requirement that it is nonzero. With the momentum operators $p_\tau = (\hbar/i)(\partial_\tau + \coth\tau)$ and $p_\varphi = (\hbar/i)\partial_\varphi$ we get for the quantum Hamiltonian (together with the quantum potential $\propto\hbar^2$)

$$H = \frac{\hbar^2}{2m}\left[p_\tau^2 + \frac{1}{\sinh^2\tau}\left(p_\varphi - \frac{eB}{\hbar c}\right)^2\right] + \frac{\hbar^2}{8m}\left(1 - \frac{1}{\sinh^2\tau}\right) \tag{22}$$

The angular variable φ varies in the interval $[0, 2\pi)$, and therefore we usually assume $\varphi_j \in [0, 2\pi)$, $\forall_j$. However, the path can loop around the infinitesimal solenoid many times, which has the consequence that in our case $\varphi_j \in \mathbb{R}$, $\forall_j$. Therefore, the path integral, if calculated according to (11), gives only a partial propagator which belongs to a class of paths topologically constrained by $\varphi_j \in [0, 2\pi)$, $\forall_j$. For the total propagator, we have to take into account all paths from all homotopically different classes. This can be done by considering the path integration over the angular variable φ_j remaining in the physical space $\mathbb{M}$ with $\Delta\varphi_j = \varphi_j - \varphi_{j-1} + 2\pi n$ ($\varphi_j \in [0, 2\pi)$, $n \in \mathbb{Z}$), or alternatively switching to the covering space $\mathbb{M}^*$ with $\Delta\varphi_j = \varphi_j - \varphi_{j-1}$, where $\varphi_j \in \mathbb{R}$. We therefore incorporate the effect of the infinitesimal thin solenoid by a δ-function constraint in the path integral, with an additional integration $\int d\varphi$ (Berndio and Inomata, 1980, 1981), and get (expanding the δ-function, $\xi = e\Phi/2\pi\hbar c$ with Φ the magnetic flux)

$$\begin{aligned}
&K^{AB}(\tau'', \tau', \varphi'', \varphi'; T)\\
&= \int_{\mathbb{R}} d\varphi \int_{\tau(0)=\tau'}^{\tau(T)=\tau''} \mathcal{D}\tau(t)\,\sinh\tau \int_{\varphi(0)=\varphi'}^{\varphi(T)=\varphi''} \mathcal{D}\varphi(t)\,\delta\left(\varphi - \int_0^T \dot\varphi\,dt\right)\\
&\quad\times \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m}{2}(\dot\tau^2 + \sinh^2\tau\,\dot\varphi^2) + \xi\dot\varphi\right.\right.\\
&\qquad\left.\left. - \frac{\hbar^2}{8m}\left(1 - \frac{1}{\sinh^2\tau}\right)\right] dt\right\}\\
&= \exp\left(-\frac{i\hbar T}{8m}\right)\int_{\mathbb{R}} d\varphi \int_{\mathbb{R}} \frac{d\lambda}{2\pi}\, e^{i\lambda\varphi} \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^N\\
&\quad\times \prod_{j=1}^{N-1} \int_0^\infty \sinh\tau_j\,d\tau_j \int_0^{2\pi} d\varphi_j\\
&\quad\times \exp\left[\frac{i}{\hbar}\sum_{j=1}^N\left(\frac{m}{2\epsilon}(\Delta^2\tau_j + \widehat{\sinh}^2\tau_j\,\Delta^2\varphi_j)\right.\right.
\end{aligned}$$

$$+ (\xi - \lambda)\Delta\varphi_j - \frac{\epsilon\hbar^2}{8m\sinh^2\tau_j}\Bigg)\Bigg]$$

$$= \int_{\mathbb{R}} d\varphi \int_{\mathbb{R}} \frac{d\lambda}{2\pi}\, e^{i\lambda\varphi} \sum_{l=-\infty}^{\infty} e^{il(\varphi''-\varphi')} K_{\lambda+l-\xi}(\tau'', \tau'; T) \tag{23}$$

where

$$\begin{aligned} &K_{\lambda+l-\xi}(\tau'', \tau'; T) \\ &\quad = e^{-i\hbar T/8m} \int_{\tau(0)=\tau'}^{\tau(T)=\tau''} \mathcal{D}\tau(t) \\ &\qquad \times \exp\left[\frac{i}{\hbar}\int_0^T \left(\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{2m}\frac{(\lambda + l - \xi)^2 - 1/4}{\sinh^2\tau}\right) dt\right] \end{aligned} \tag{24}$$

Using Poisson's summation formula

$$\sum_{l=-\infty}^{\infty} e^{il\theta} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\theta + 2\pi k) \tag{25}$$

we obtain (by changing the integration variable $\lambda \to \lambda + \xi - l$)

$$\begin{aligned} &K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} d\varphi \int_{\mathbb{R}} d\lambda\, e^{i\lambda\varphi} \sum_{l=-\infty}^{\infty} e^{il(\varphi''-\varphi')} K_{\lambda+l-\xi}(\tau'', \tau'; T) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} d\varphi \int_{\mathbb{R}} d\lambda\, e^{il(\varphi''-\varphi'-\varphi)+i(\lambda+\xi)\varphi} K_\lambda(\tau'', \tau'; T) \\ &= \int_{\mathbb{R}} d\varphi \sum_{k=-\infty}^{\infty} \delta(\varphi'' - \varphi' - \varphi + 2\pi k) e^{i(\lambda+\xi)\varphi} \int_{\mathbb{R}} d\lambda\, K_\lambda(\tau'', \tau'; T) \end{aligned} \tag{26}$$

K_λ is now given by

$$\begin{aligned} &K_\lambda(\tau'', \tau'; T) \\ &\quad = e^{-i\hbar T/8m} \int_{\tau(0)=\tau'}^{\tau(T)=\tau''} \mathcal{D}\tau(t) \exp\left[\frac{i}{\hbar}\int_0^T \left(\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\sinh^2\tau}\right) dt\right] \\ &\quad = \int_0^\infty dk\, e^{-iE_kT/\hbar}\, \Psi_{k,\lambda}(\tau'')\Psi^*_{k,\lambda}(\tau') \end{aligned} \tag{27}$$

The wavefunctions and the energy spectrum are given by (17) and (18), respectively, with $l \to \lambda$. Performing the φ integration in (26) yields

$$K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) = \sum_{n=-\infty}^{\infty} e^{i\xi(\varphi''-\varphi'+2\pi n)} \int_{\mathbb{R}} d\lambda\, e^{i\lambda(\varphi''-\varphi'+2\pi n)} K_\lambda(\tau'', \tau'; T) \tag{28}$$

which displays the expansion in the winding numbers. For $\xi = 0$ the free Feynman kernel on Λ is recovered.

If we want to study the effect of scattering by an Aharonov–Bohm solenoid we must consider interference terms according to

$$I_{nl} = K_n^* K_l + K_l^* K_n \tag{29}$$

Unfortunately, a closed expression for the propagator (27) does not exist. We can either analyze (27) by means of an asymptotic expansion of the Legendre functions, i.e.,

$$P^{\mu}_{ip-1/2}(z) \propto [\Gamma(ip)/\Gamma(1/2 + ip - \mu)](2z)^{1/2-ip}/\sqrt{\pi} + \text{c.c.} \qquad \text{as} \quad |z| \to \infty$$

which yields very complicated and analytically intractable integrals over Γ functions. Alternatively, we can use the formula $\lim_{\nu\to\infty} \nu^{\mu}\mathcal{P}_\nu^{-\mu}(\cosh(z/\nu)) = I_\mu(z)$ (Gradshteyn and Ryzhik, 1980), which corresponds to the flat-space limit of the hyperbolic space with constant curvature R. Restricting therefore the evaluation of I_{nl} to the flat-space limit $R \to \infty$, we reintroduce the constant curvature R into the path integral (27) by means of $m\dot{\tau}^2 \to mR^2\dot{\tau}^2 = mr^2$, and $m \sinh^2\tau \to mR^2 \sinh^2\tau \to mR^2\tau^2 = mr^2$ ($r = R\tau$ is the radial variable in Euclidean polar coordinates) as $R \to \infty$ (Izmest'ev *et al.*, 1997). This gives for K_λ in this limit the usual free Feynman kernel in polar coordinates in $\mathbb{R}^2$ (Grosche and Steiner, 1998; Peak and Inomata, 1969)

$$K_\lambda(\tau'', \tau'; T) \simeq K_\lambda(r'', r'; T) = \frac{m}{2\pi i\hbar T} \exp\left[\frac{im}{2\hbar T}(r'^2 + r''^2)\right] I_{|\lambda|}\left(\frac{mr'r''}{i\hbar T}\right) \tag{30}$$

Following Berndio and Inomata (1980, 1981), we can now evaluate I_{nl}. By means of the asymptotic formula $[|z| \to \infty, \Re(z) > 0]$

$$I_\lambda(z) \simeq \sqrt{\frac{1}{2\pi z}} \exp\left(z - \frac{\lambda^2 - 1/4}{2z}\right) \tag{31}$$

and a Gaussian integration we get the asymptotic expansion

$$\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda\Theta} I_\lambda(z) \simeq \exp\left(z + \frac{1}{8z} - \frac{z}{2}\Theta^2\right) \tag{32}$$

Hence we obtain for the partial propagator K_n (with $z = mr'r''/i\hbar T$, ignoring the condition $\Re(z)$ (Berndio and Inomata, 1980, 1981; Grosche and Steiner, 1998; Peak and Inomata, 1969),

$$K_n(\tau'', \tau', \varphi'', \varphi'; T)$$

$$\simeq \frac{m}{2\pi i\hbar T} \exp\Bigg[\frac{imR^2}{2\hbar T} (\tau'' - \tau')^2 + \frac{i\hbar T}{8mR^2\tau'\tau''} + i\xi(\varphi'' - \varphi' + 2\pi n) + \frac{imR^2\tau'\tau''}{2\hbar T} (\varphi'' - \varphi + 2\pi n) \Bigg] \tag{33}$$

Consequently, we get for the interference term

$$I_{nl} \simeq 2\left(\frac{m}{2\pi i\hbar T} \right)^2 \times \cos\Bigg[2\pi(l - n)\left(\xi + \frac{mR^2\tau'\tau''}{\hbar T} (\varphi'' - \varphi' - \pi) \right) + 2\pi^2 \frac{mR^2\tau'\tau''}{\hbar T} (l - n)(l + n + 1) \Bigg] \tag{34}$$

The principal feature of this result is that the interference patterns do not depend only on the initial (τ', φ') and final points (τ'', φ''), but on the homotopy class numbers n and m as well, which describe the windings around the infinitesimal thin solenoid. This flux-dependent shift is a proper Aharonov–Bohm effect. The interference term vanishes for $n = l$.

The maximum contribution to the Aharonov–Bohm effect on the (hyperbolic) plane is observed for the smallest nonvanishing value $|n - l| = 1 > 0$. Therefore, the maximum effect is observed for the interference of the winding number $l = 0$ and $n = -1$, or vice versa, yielding the interference term

$$I_{0,-1} = 2\left(\frac{m}{2\pi i\hbar T} \right)^2 \cos(2\pi\xi) \tag{35}$$

This is the standard result; see, e.g., Feynman and Hibbs (1965) and Berndio and Inomata (1980, 1981), and references therein.

3. HIGGS OSCILLATOR AND KEPLER–COULOMB POTENTIAL

Obviously, we can incorporate potential terms in the radial path integration τ, e.g., we can include the Higgs-oscillator potential (Grosche *et al.*, 1996; Higgs, 1979)

$$V_{(\mathrm{Higgs})}(\mathbf{u}) = \frac{m}{2}\omega^2 R^2 \frac{u_1^2 + u_2^2}{u_0^2} = \frac{m}{2}\omega^2 R^2 \tanh^2\tau \tag{36}$$

which is the analogue of the harmonic oscillator in a space of constant curvature, or the Kepler–Coulomb potential (Barut *et al.*, 1990; Grosche, 1990b; Grosche *et al.*, 1996),

$$V_{(\mathrm{Coulomb})}(\mathbf{u}) = -\frac{\alpha}{R}\left(\frac{u_0}{\sqrt{u_1^2 + u_2^2}} - 1\right) = -\frac{\alpha}{R}(\coth\tau - 1) \tag{37}$$

For clarity, I have included the dependence on the constant curvature R explicitly. In these cases, the result (23) is more appropriate. The combined $d\varphi d\lambda$ integration yields $\lambda = 0$, and the total propagator becomes

$$K^{AB}(\tau'', \tau', \varphi'', \varphi'; T) = \sum_{l=-\infty}^{\infty} e^{il(\varphi''-\varphi')} K_{|l-\xi|}(\tau'', \tau', T) \tag{38}$$

and the effect of the solenoid results in a modification of the angular momentum dependence of $K_{|l-\xi|}$. This feature, however, modifies the number of bound states of the system with respect to the quantum number l. For instance, for the Higgs-oscillator case this gives ($\nu^2 = m^2\omega^2 R^4/\hbar^2 + 1/4$)

$$\Psi_{nl}^{(\mathrm{Higgs})}(\tau, \varphi; R) = (2\pi\sinh\tau)^{-1/2} S_n^{(\nu)}(\tau; R) e^{il\varphi} \tag{39}$$

$$\begin{aligned} S_n^{(\nu)}(\tau; R) &= \frac{1}{\Gamma(|l-\xi| + 1)} \\ &\times \left[\frac{2(\omega - |l-\xi| - 2n - 1)\Gamma(n + |l-\xi| + 1)\Gamma(\nu - |l-\xi|)}{R^2\Gamma(\nu - |l-\xi| - n)n!}\right]^{1/2} \\ &\times (\sinh\tau)^{|l-\xi|+1/2}(\cosh\tau)^{n+1/2-\nu} \\ &\times {}_2F_1(-|l-\xi|, \nu - n; 1 + |l-\xi|; \tanh^2\tau) \end{aligned} \tag{40}$$

with the discrete spectrum given by

$$E_n^{(\mathrm{Higgs})} = -\frac{\hbar^2}{2mR^2}\left[(2n + |l-\xi| - \nu + 1)^2 - \frac{1}{4}\right] + \frac{m}{2}\omega^2 R^2 \tag{41}$$

Only a finite number exist with $N_{\max} = [\nu - |l-\xi| - 1] \geq 0$ ($[x]$ denotes the integer value of $x \in \mathbb{R}$). The continuous wavefunctions have the form

$$\Psi_{kl}^{(\text{Higgs})}(\tau, \varphi; R) = (2\pi \sinh \tau)^{-1/2} S_k^{(\nu)}(\tau; R) e^{il\varphi} \tag{42}$$

$$\begin{aligned} S_k^{(\nu)}(\tau; R) = & \frac{1}{\Gamma(|l-\xi|+1)} \sqrt{\frac{k \sinh \pi k}{2\pi^2 R^2}} \\ & \times \Gamma\left(\frac{\nu - |l-\xi| + 1 - ik}{2}\right) \Gamma\left(\frac{|l-\xi| - \nu + 1 - ik}{2}\right) \\ & \times (\tanh \tau)^{|l-\xi|+1/2} (\cosh \tau)^{ik} \qquad (43) \\ & \times {}_2F_1\left(\frac{\nu + |l-\xi| + 1 - ik}{2}, \frac{|l-\xi| - \nu + 1 - ik}{2}; \right. \\ & \left. 1 + |l-\xi|; \tanh^2 \tau\right) \qquad (44) \end{aligned}$$

with the continuous energy spectrum given by

$$E_p^{(\text{Higgs})} = \frac{\hbar^2}{2mR^2}\left(k^2 + \frac{1}{4}\right) + \frac{m}{2}\omega^2 R^2 \tag{45}$$

In the case of the Kepler–Coulomb problem on Λ we obtain for the discrete energy spectrum ($\tilde{N} = N + |l - \xi| + \frac{1}{2}$; $N = 0, 1, 2, \ldots, N_{\max} = [\sqrt{R/a} - |l - \xi| - \frac{1}{2}]$; $a = \hbar^2/m\alpha$ is the Bohr radius)

$$E_N^{(\text{Coulomb})} = \frac{\alpha}{R} - \hbar^2 \frac{\tilde{N}^2 - \frac{1}{4}}{2mR^2} - \frac{m\alpha^2}{2\hbar^2 \tilde{N}^2} \tag{46}$$

I do not state the wavefunctions here (Grosche *et al.*, 1996), and the continuous states are modified by their angular momentum dependence, i.e., $l \to l - \xi$. However, the effect of the Aharonov–Bohm field is not restricted to a modification of the discrete spectrum, but the effect on the scattering states happens through an interference term I_{nl} similar to (29), for the Coulomb potential and the Higgs oscillator as well. Again, a closed expression for the radial propagator does not exist and we are restricted to the investigation of the limiting case along the lines following (29). I do not repeat this here.

4. SUMMARY

I have shown the admissibility of path integration of the Aharonov–Bohm effect on the hyperbolic plane. It can be studied in a straightforward manner yielding analogous results to the flat-space case. For scattering states we find interference, due to the modification of the angular momentum dependence according to $l \to l - \xi$, giving a cos-like pattern in terms of the strength of the vector potential for the free motion, the Kepler–Coulomb problem, and the Higgs oscillator (which is absent in the flat-space case); the bound-state wavefunction and the corresponding energy levels are modified in their angular momentum dependence $l \to l - \xi$ as well, including an alteration of the number of bound states. We found the usual expansion of the total propagator in terms of an expansion in the winding number n of the homotopy class of paths. All these features are well-known from the corresponding flat-space cases. The complicated interference expression (29) could not be evaluated due the non-constant-curvature features of the hyperbolic plane. This would involve an intractable analytical integration over Legendre functions with respect to the order. However, the investigation of the flat-space limit gave the well-known result. Therefore the effect of an Aharonov–Bohm gauge field on the hyperbolic plane, i.e., scattering on leaky tori, exhibits the same features as in the flat-space case of $\mathbb{R}^2$.

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